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THE TRACE-CLASS OF A FULL HILBERT ALGEBRA

BY

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ABSTRACT. The trace-class of a full Hilbert algebra A is the set $\tau(A) = \{xy \mid x \in A, y \in A\}$. This set is shown to be a $*$-ideal of A, and possesses a norm τ defined in terms of a positive hermitian linear functional on $\tau(A)$. The norm τ is in general both incomplete and not an algebra norm, and is also not comparable with the Hilbert space norm $\| \ \|$ on $\tau(A)$. However, a one-sided ideal of $\tau(A)$ is closed with respect to one norm if and only if it is closed with respect to the other. The topological dual of $\tau(A)$ with respect to the norm τ is isometrically isomorphic to the set of left centralizers on A.

Introduction. The methods of Schatten [10], employed by Saworotnow and Friedell [8] in the H^*-algebra setting, are used here in §§1 and 2, to show that the trace-class $\tau(A)$ of a full Hilbert algebra A is a *-ideal of A (Theorem 2.2), and to define a norm τ and a positive hermitian linear functional on $\tau(A)$. These enjoy many of the same properties as for H^*-algebras, with some exceptions (Theorem 2.5): the norm τ is generally incomplete and is not an algebra norm on $\tau(A)$, unless A itself is complete, in which case A is an H^*-algebra in a trivially equivalent norm.

§3 deals with two theorems concerning the trace-class (see [10, §IV.1], and also [9] for the H^*-algebra setting). Theorem 3.1 shows that the topological dual of $\tau(A)$ is isometrically isomorphic to the set of left centralizers on A, while Theorem 3.2 says that $\tau(A)$ is isometrically isomorphic to a subspace of $C^*(A)$, the C^*-algebra of A. An example is given (3.3) to show that this subspace may not even be dense in $C^*(A)$.

In §4 we examine the relation between the two norms τ and $\| \ \|$ on the trace-class. They are in general incomparable (Theorem 2.5 and 4.1). However, in Theorem 4.5 it is shown that a one-sided ideal of $\tau(A)$ is closed in one norm topology if and only if it is closed in the other. It is noted (Theorem 4.7) that $\tau(A)$ is an orthocomplemented Hilbert algebra. The final result is that the closed ideals of $\tau(A)$ are precisely the trace-classes of closed ideals of A. Many of these results

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are generalizations of the same occurrences for H^*-algebras, as discovered by Smith in [11].

1. **Basic results.** Let A denote a Hilbert algebra with inner product (x, y), norm $\|x\| = (x, x)^{1/2}$, and involution $x \to x^*$. The definition and elementary properties of a Hilbert algebra can be found in [1] and [7]; see also [2], [6] and [12]. Our notation follows that of Yood in [14]. The Hilbert space which is the completion of A in the norm $\|x\|$ is denoted H, and A_b is the fulfillment of A. $\mathcal{B}(H)$ shall denote the space of all bounded linear operators on H.

Put $\Lambda(A) = \{\overline{R}_y: y \in A\}'$ (the commutant taken in $\mathcal{B}(H)$) and $P(A) = \{\overline{L}_x: x \in A\}'$. The commutation theorem (see [2] or [7]) states that $\Lambda(A)' = P(A)$ and $P(A)' = \Lambda(A)$. Moreover, $\Lambda(A)$ (resp. $P(A)$) is the strong closure of $\{\overline{L}_x: x \in A\}$ (resp. of $\{\overline{R}_y: y \in A\}$), and $\Lambda(A) = \Lambda(A_b)$, $P(A) = P(A_b)$. Since A_b is invariant under $\Lambda(A_b) \cup P(A_b)$ [7, Proposition 1.6], $\Lambda(A_b)$ may also be defined as the set of operators in $\mathcal{B}(H)$ satisfying $T(xy) = T(x)y$, $\forall x$ and y in A_b. We call such operators left centralizers on A_b after B. E. Johnson [4]; operators in $P(A_b)$ are called right centralizers on A_b. Johnson's terminology differs (i.e. "left" in place of "right") from other notions of centralizer found, for example, in [13] and [5].

As in [8], a projection shall be a self-adjoint idempotent, and a projection base for A shall be a maximal family of nonzero mutually orthogonal projections of A. A Hilbert algebra in general need not contain any nonzero projections (see [14, §4]), but projection bases exist in full Hilbert algebras (see [7, Theorem 2.3]). Thus we restrict our attention to the latter; in the remainder of the paper, A shall denote a full Hilbert algebra.

We need two results of Rieffel [7, Theorems 3.8 and 3.9], but enlarged to involve projection bases.

1.1 **Lemma.** *Let $T \in \Lambda(A)$ (resp. $T \in P(A)$). The following statements are equivalent*:

(1) $T = \overline{L}_x$ *(resp. $T = \overline{R}_x$) for some x in A.*

(2) $\sup\{\|Te\|: e$ *is a projection of* $A\} < +\infty$.

(3) *There is a projection base $\{e_\gamma: \gamma \in \Gamma\}$ for A such that $\sum_{\gamma \in \Gamma}\|Te_\gamma\|^2 < +\infty$.*

1.2 **Lemma.** *Let $a \in A$. The following statements are equivalent*:

(1) *a is positive and integrable.*

(2) *a is positive and $\sum_{\gamma \in \Gamma}(a, e_\gamma) < +\infty$ for some projection base $\{e_\gamma: \gamma \in \Gamma\}$ for A.*

(3) *$a = b^2$ for some unique positive b in A.*

(4) *$a = x^*x$ for some x in A.*

Proof. Clearly (3) $\Rightarrow$ (4) $\Rightarrow$ (1) $\Rightarrow$ (2). If (2) holds, and T is the unique positive square root in $\Lambda(A)$ of the positive operator $\overline{L}_a$, then

$$\sum_{\gamma\in\Gamma} \|Te_\gamma\|^2 = \sum_{\gamma\in\Gamma} (T^2 e_\gamma, e_\gamma) = \sum_{\gamma\in\Gamma} (ae_\gamma, e_\gamma) < +\infty,$$

so $T = \overline{L}_b$ for some positive b in A by Lemma 1.1. Since the mapping $x \to \overline{L}_x$ is an algebra *-isomorphism on A, it follows that b is the unique positive square root of a. Thus (2) $\Rightarrow$ (3).

Using notation of Schatten (see [10] and [8]), we let $[x]$ denote the positive square root in A of x^*x, for each x in A. Note that $\|[x]\| = \|x\|$, and $\overline{L}_{[x]} = [\overline{L}_x]$, the positive square root of $\overline{L}_{x^*x}$.

It is interesting to observe that a complete analogue of the polar decomposition theorem for operators in $\mathcal{B}(H)$ (see [10]) obtains in the full Hilbert algebra setting. The partial isometry involved does not have as exact a description as for $\mathcal{B}(H)$ or any H^*-algebra (see [8]), however that is an unnecessary detail.

1.3 **Theorem.** *For each x in A, there is a partial isometry W_x in $\Lambda(A)$ with initial set $\overline{[x]A}^H$ (the closure in H of $[x]A$) and final set $\overline{xA}^H$ such that*

(1) $x = W_x([x])$;

(2) $[x] = W_x^*(x)$;

(3) $x^* = W_x^*([x^*])$;

(4) $[x^*] = W_x(x^*)$.

Moreover, if $x = W(b)$ for some positive b in A and partial isometry W in $\Lambda(A)$ with initial set $\overline{bA}^H$, then $b = [x]$ and $W = W_x$.

Proof. Use the polar decomposition theorem in [10] to obtain a partial isometry W_x in $\mathcal{B}(H)$ with initial set $\overline{[x]A}^H$ and final set $\overline{xA}^H$ such that $\overline{L}_x = W_x[\overline{L}_x] = W_x\overline{L}_{[x]}$, etc. For convenience let $\mathfrak{N}$ denote the orthogonal complement in H of $\overline{[x]A}^H$. $\mathfrak{N}$ is invariant under each $\overline{R}_y$ (y in A) so $\overline{R}_y W_x$ and $W_x \overline{R}_y$ agree on $H = \overline{[x]A}^H \oplus \mathfrak{N}$, for each y in A. Thus W_x is a left centralizer. (1)–(4) now follow using the semisimplicity of A.

If $x = W(b)$ as in the last sentence of the theorem, then

$$\overline{L}_{x^*x} = \overline{L}_x^{\,*}\overline{L}_x = \overline{L}_b W^* W \overline{L}_b = \overline{L}_b^2 = \overline{L}_{b^2},$$

so $b = [x]$. If follows that W and W_x agree on $\overline{[x]A}$, hence are equal.

As one might expect, there is a parallel result concerning right centralizers.

1.4 **Theorem.** *For each x in A, there is a partial isometry V_x in $P(A)$ with initial set $\overline{A[x]}^H$ and final set $\overline{Ax^*}^H$, such that*

(1) $x^* = V_x([x])$;

(2) $[x] = V_x^*(x^*)$;

(3) $x = V_x{}^*([x^*])$;

(4) $[x^*] = V_x(x)$.

If $x^* = V(b)$ *for some positive* b *in* A *and partial isometry* V *in* $P(A)$ *with initial set* $\overline{Ab^H}$, *then* $b = [x]$ *and* $V = V_x$.

Suppose for any S in $\mathcal{B}(H)$ we define an operator $S^\#$ on H by $S^\#(\xi) = S(\xi^*)^*$, ξ in H. The mapping $S \to S^\#$ is a conjugate-linear isometric automorphism of period 2 of $\mathcal{B}(H)$ onto itself, with the following properites:

(a) $\overline{L}_x^\# = \overline{R}_{x^*}$, for any x in A [2, Lemma 3];

(b) $\Lambda(A)^\# = P(A)$, $P(A)^\# = \Lambda(A)$;

(c) if $P_{\mathfrak{M}}$ is the projection of H onto a closed subspace $\mathfrak{M}$, then $P_{\mathfrak{M}}^\# = P_{\mathfrak{M}^*}$;

(d) if U is a partial isometry with initial set $\mathfrak{M}$ and final set $\mathfrak{N}$, then $U^\#$ is a partial isometry with initial set $\mathfrak{M}^*$, final set $\mathfrak{N}^*$;

(e) $^\#$ commutes with *, the adjoint operation on $\mathcal{B}(H)$.

From this it follows that $V_x = W_x^\#$, $W_{x^*} = W_x^*$, and $V_{x^*} = V_x^*$ for each x in A.

2. **The trace-class.** The trace-class of A is the set $\tau(A) = \{xy: x \in A, y \in A\}$. This set is not obviously closed under addition. To show that this is so, we emulate the procedure in [10] and, more exactly, in [8]. To begin with, every element in the trace-class is integrable. It is not clear whether the converse obtains—however, it does for positive elements.

2.1 **Lemma.** *For any* a *in* A, *the following statements are equivalent*:

(1) a *is in* $\tau(A)$.

(2) $[a]$ *is in* $\tau(A)$.

(3) $[a]$ *is integrable.*

(4) *There is a projection base* $\{e_\gamma: \gamma \in \Gamma\}$ *for* A *such that* $\sum_{\gamma \in \Gamma}([a], e_\gamma) < +\infty$.

(5) $[a]$ *has a unique positive square root* $[a]^{1/2}$ *in* A.

Proof. Use 1.2 and Theorem 1.3.

For any x and y in A and any projection base $\{e_\gamma: \gamma \in \Gamma\}$ for A, the sum $\sum_{\gamma \in \Gamma}(xy, e_\gamma)$ converges absolutely to the number (x, y^*), and is therefore independent of the choice of projection base. This number is called the trace of xy, $\mathrm{tr}(xy)$: $\mathrm{tr}(a) = \sum_{\gamma \in \Gamma}(a, e_\gamma)$ for any a in $\tau(A)$ and projection base $\{e_\gamma : \gamma \in \Gamma\}$ for A.

2.2 **Theorem.** $\tau(A)$ *is a dense* **-ideal of* A *which is invariant under left or right centralizers.* tr *is a positive hermitian linear functional on* $\tau(A)$ *such that*

(1) $\mathrm{tr}(xy) = \mathrm{tr}(yx) = (x, y^*)$,

(2) $\mathrm{tr}(x^*x) = \|x\|^2$,

for any x *and* y *in* A.

Proof. It is clear that $\tau(A)$ is invariant under left or right centralizers. The proof that $\tau(A)$ is closed under addition is similar to Schatten's [10, Lemma 3, p. 38]. The rest of the theorem now follows easily.

Now define $\tau(a) = \mathrm{tr}([a])$ for each a in $\tau(A)$. Then $\tau(a) = \Sigma_{\gamma \in \Gamma}([a], e_\gamma)$ for each projection base $\{e_\gamma : \gamma \in \Gamma\}$ for A. Right away we see that

$$\tau(a) = \tau([a]) = \|[a]^{1/2}\|^2$$

and $\tau(\lambda a) = \mathrm{tr}(|\lambda|[a]) = |\lambda|\tau(a)$, for any a in $\tau(A)$ and complex number λ. τ will be a norm on the trace-class once we show it is subadditive, and so we come to the next result (see [10, p. 39] as well as [8]).

2.3 **Lemma.** *For any a in $\tau(A)$ and operator T in $\Lambda(A) \cup P(A)$,*

(1) $|\mathrm{tr}(Ta)| \leq \|T\|\tau(a)$,

(2) $\tau(Ta) \leq \|T\|\,\tau(a)$.

Proof. If T is a left centralizer, the proof is the same as in [8]. If T is a right centralizer, we proceed slightly differently:

$$\begin{aligned} |\mathrm{tr}(Ta)| &= |\mathrm{tr}(TW_a([a]^{1/2}[a]^{1/2}))| = |\mathrm{tr}\; T(W_a([a]^{1/2})[a]^{1/2})| \\ &= |\mathrm{tr}\; W_a([a]^{1/2})T([a]^{1/2})| = |(W_a[a]^{1/2}, (T[a]^{1/2})^*)| \\ &\leq \|W_a[a]^{1/2}\|\,\|T[a]^{1/2}\| \leq \|[a]^{1/2}\|\,\|T\|\,\|[a]^{1/2}\| = \|T\|\tau(a). \end{aligned}$$

Since T commutes with operators in $\Lambda(A)$, we have

$$\begin{aligned} \tau(Ta) &= \mathrm{tr}([Ta]) = \mathrm{tr}(W_{Ta}{}^*(Ta)) = \mathrm{tr}(TW_{Ta}{}^*(a)) \\ &\leq \|T\|\tau(W_{Ta}{}^*(a)) \leq \|T\|\,\|W_{Ta}{}^*\|\tau(a) \leq \|T\|\tau(a). \end{aligned}$$

2.4 **Theorem.** *τ is a linear space norm on $\tau(A)$ with the following properties:*

(1) *multiplication in $\tau(A)$ is separately τ-continuous;*

(2) $\tau(a^*) = \tau(a)$ *for each a in $\tau(A)$;*

(3) $|\mathrm{tr}\; a| \leq \tau(a)$ *for each a in $\tau(A)$;*

(4) $\tau(xy) \leq \|x\|\,\|y\|$ *for every x and y in A;*

(5) $\tau(T) = \|T\|$ *for every T in $\Lambda(A) \cup P(A)$.*

Proof. τ is subadditive: for any a and b in $\tau(A)$, we have

$$\begin{aligned} \tau(a+b) &= \mathrm{tr}([a+b]) = \mathrm{tr}(W^*_{a+b}(a) + W^*_{a+b}(b)) \\ &\leq |\mathrm{tr}(W^*_{a+b}(a))| + |\mathrm{tr}(W^*_{a+b}(b))| \leq \tau(a) + \tau(b), \end{aligned}$$

using Theorem 1.3 and Lemma 2.3. Thus τ is a linear space norm on the trace-class. (1) follows from Lemma 2.3 also, as does (3). To prove (2), we have, using Theorem 1.4 and Lemma 2.3,

$$\tau(a^*) = \tau(V_a([a])) \leq \|V_a\|\tau([a]) = \tau(a)$$

for each a in $\tau(A)$, so equality obtains. (4) is proven as in [8, Corollary 4]. If T is a left or right centralizer on A, its restriction to the normed linear space $\tau(A)$ is continuous with respect to the norm τ by Lemma 2.3. The norm of the restricted operator is denoted $\tau(T)$:

$$\tau(T) = \sup\{\tau(Ta): a \in \tau(A) \text{ and } \tau(a) \leq 1\}.$$

Now $\tau(T) \leq \|T\|$ by 2.3. We prove the reverse inequality for left centralizers (proof is similar for right centralizers): for any x in A,

$$\begin{aligned}\|Tx\|^2 &= \|(Tx)^*\|^2 = \tau(Tx(Tx)^*) \\ &= \tau(T(x(Tx)^*)) \leq \tau(T)\tau(x(Tx)^*) \leq \tau(T)\|x\|\|Tx\| \quad \text{by (4) above;}\end{aligned}$$

thus $\|Tx\| \leq \tau(T)\|x\|$ for each x in A. Thus $\|T\| \leq \tau(T)$, so equality obtains.

Thus far the trace-class $\tau(A)$ and its norm τ have behaved much the same as in the H^*-algebra setting. Now however we notice some differences: τ is not an algebra norm on $\tau(A)$, and is incomplete. One may attribute these failings to the lack of the same properties of the norm $\| \ \|$ on A, as we see from the next result.

2.5 **Theorem.** *The following statements are equivalent:*

(1) *Multiplication in* $\tau(A)$ *is jointly* τ*-continuous.*

(2) *There is a constant* $M > 0$ *such that* $\tau(ab) \leq M\tau(a)\tau(b)$ *for every* a *and* b *in* $\tau(A)$.

(3) *There is a constant* $K > 0$ *such that* $\|a\| \leq K\tau(a)$ *for each* a *in* $\tau(A)$.

(4) τ *is a complete norm on* $\tau(A)$.

(5) $\| \ \|$ *is a complete norm on* A *(so* $A = H$*).*

(6) *Multiplication in* A *is jointly continuous.*

(7) A *is trivially renormable to be an* H^**-algebra.*

Proof. (5), (6), and (7) are equivalent by Lemma 4.5 of [14]. The equivalence of (1) and (2) is a simple matter. If (2) is true, then for each a in $\tau(A)$,

$$\|a\|^2 = \tau(a^*a) \leq M\tau(a^*)\tau(a) = M\tau(a)^2$$

so (3) holds. If (3) is true, then so is (6): for any x and y in A,

$$\|xy\| \leq K\tau(xy) \leq K\|x\|\|y\| \quad \text{(using (4) of Theorem 2.4).}$$

Suppose now that A is trivially renormable to be an H^*-algebra, and suppose the H^*-algebra norm on A is $\|x\|_1 = c\|x\|$ (x in A). Then $(x, y)_1 = c^2(x, y)$ for all x and y in A, so $\tau_1(a) = c^2\tau(a)$ for all a in $\tau(A) = \tau_1(A)$. But $\tau_1(A) = \tau(A)$ is a Banach algebra in the norm τ_1 (see [8] and [9]), hence τ itself is complete on the trace-class. Thus (7) $\Rightarrow$ (4). Finally, suppose (4) is true; then Lemma 2.3 implies that

$$\sup\{\tau(\bar{L}_a(b)) : a \in \tau(A),\ \tau(a) \le 1\} \le \|\bar{R}_b\| < +\infty$$

for each b in $\tau(A)$. An application of the uniform boundedness principle gives

$$M \equiv \sup\{\tau(\bar{L}_a) : a \in \tau(A),\ \tau(a) \le 1\} < +\infty.$$

It follows that $\tau(ab) = \tau((a/\tau(a)) \cdot \tau(a)b) \le M\tau(\tau(a)b) = M\tau(a)\tau(b)$ for all $a \ne 0$ in $\tau(A)$ and b in $\tau(A)$; thus (2) is true. This completes the proof of the theorem.

3. **The dual of the trace-class.** What follows now is an attempt to extend two results of Schatten (see [10, pp. 46–48], as well as [9, Theorems 1 and 2] for the H^*-algebra case). One extends fully, the other only partially. We use the following notation: if Ψ is a linear functional on $\tau(A)$ which is continuous with respect to the norm τ, we let $\tau(\Psi)$ denote the sup norm of Ψ:

$$\tau(\Psi) = \sup\{|\Psi(a)| : a \in \tau(A) \text{ and } \tau(a) \le 1\}.$$

$\tau(A)'$ shall denote the set of all τ-continuous linear functionals on $\tau(A)$. For example, $\mathrm{tr} \in \tau(A)'$ and $\tau(\mathrm{tr}) = 1$.

3.1 **Theorem.** *For T in $\Lambda(A)$, define a functional Ψ_T on $\tau(A)$ by $\Psi_T(a) = \mathrm{tr}(Ta)$ (a in $\tau(A)$). The mapping $T \to \Psi_T$ is a linear isometry of $\Lambda(A)$ onto $\tau(A)'$: $\tau(\Psi_T) = \|T\|$.*

Proof. See [9, Theorem 2].

Let $C^*(A)$ denote the C^*-algebra of A, the operator norm closure in $\mathcal{B}(H)$ (or in $\Lambda(A)$) of the space $\{\bar{L}_x : x \in A\}$.

3.2 **Theorem.** *For a in $\tau(A)$, define a functional ϕ_a on $C^*(A)$ by $\phi_a(T) = \mathrm{tr}(Ta)$ (T in $C^*(A)$). The mapping $a \to \phi_a$ is a linear isometry of $\tau(A)$ into the space of continuous linear functionals on $C^*(A)$: $\|\phi_a\| = \tau(a)$.*

Proof. By Lemma 2.3, ϕ_a is a continuous linear functional on $C^*(A)$ and $\|\phi_a\| \le \tau(a)$. The mapping $a \to \phi_a$ is clearly linear, so it remains only to show that $\tau(a) \le \|\phi_a\|$ for each a in $\tau(A)$. Use the Kaplansky density theorem (see [1, p. 46]) to obtain a sequence $\{z_n\}$ in A with $\|\bar{L}_{z_n}\| \le 1$, for all n, such that W_a^* is the limit in the strong operator topology of $\bar{L}_{z_n}$. By Theorem 1.3, $[a] = \lim_{n\to\infty} z_n a$. Now let $\{e_\gamma : \gamma \in \Gamma\}$ be any projection base for A, and let F be any finite subset of Γ. Put $p = \Sigma_{\gamma \in F}\, e_\gamma$, a projection in A. Since $\|\bar{L}_{pz_n}\| \le \|\bar{L}_p\|\,\|\bar{L}_{z_n}\| \le 1$, we have

$$\|\phi_a\| \ge |\phi_a(\bar{L}_{pz_n})| = |\mathrm{tr}(pz_n a)| = |(z_n a, p)|$$

for each n. Letting $n \to \infty$, we have $\|\phi_a\| \ge |([a], p)| = \Sigma_{\gamma \in F}\, ([a], e_\gamma)$. Since

F is an arbitrary finite subset of Γ, this means that $\|\phi_a\| \geq \Sigma_{\gamma \in \Gamma}([a], e_\gamma) = \tau(a)$, thus proving the theorem.

The difference between Theorem 3.2 and Theorem 1 if [9] is this: the image ϕ_A of the mapping $a \to \phi_a$ (a in $\tau(A)$) need not be all of the dual of $C^*(A)$. Of course this cannot be so unless τ is a complete norm on the trace-class, which means that A would have to be an H^*-algebra (after trivial renorming), by Theorem 2.5. However, ϕ_A need not even be dense in the dual of $C^*(A)$, as the following example shows:

3.3 **Example.** The notation for this example is that of [3]—see especially §§9, 10, 13, 19 and 20. Let X denote an arbitrary nonvoid locally compact Hausdorff space and $(X, \mathfrak{M}_l, l)$ a measure space of the kind discussed in [3, §§9, 10]. The measure l need not be σ-finite. For convenience, put $\mathfrak{L}_p = \mathfrak{L}_p(X, \mathfrak{M}_l, l)$, for $1 \leq p \leq \infty$. All functions considered are $\mathfrak{M}_l$-measurable. Then $\mathfrak{L}_2 \cap \mathfrak{L}_\infty$ is a commutative full Hilbert algebra under pointwise operations, the $\mathfrak{L}_2$ inner product, and conjugation as involution. For any f in $\mathfrak{L}_2 \cap \mathfrak{L}_\infty$, $\overline{L}_f(g) = fg$, $\forall g$ in $\mathfrak{L}_2$. Moreover, for any h in $\mathfrak{L}_\infty$, $hg \in \mathfrak{L}_2$ for each g in $\mathfrak{L}_2$ and $\|hg\|_2 \leq \|g\|_2 \|h\|_\infty$. Consequently, if $h \in \mathfrak{L}_\infty$, we shall write $\overline{L}_h(g) = hg$, $\forall g$ in $\mathfrak{L}_2$, noting that $\overline{L}_h \in \mathfrak{B}(\mathfrak{L}_2)$ and in fact $\|\overline{L}_h\| = \|h\|_\infty$. From this, one sees that the C^*-algebra of $\mathfrak{L}_2 \cap \mathfrak{L}_\infty$ is $\mathfrak{L}_\infty$. Using Theorem 19.30 of [3] to construct a projection base for $\mathfrak{L}_2 \cap \mathfrak{L}_\infty$, one can show that $\operatorname{tr}(f) = \int_X f\,dl$, $\forall f$ in $\tau(\mathfrak{L}_2 \cap \mathfrak{L}_\infty)$; thus $\tau(\mathfrak{L}_2 \cap \mathfrak{L}_\infty) = \mathfrak{L}_1 \cap \mathfrak{L}_\infty$ and the trace-norm is the $\mathfrak{L}_1$ norm.

If the mapping $a \to \phi_a$ of Theorem 3.2 sent $\mathfrak{L}_1 \cap \mathfrak{L}_\infty$ onto a dense subset of the dual of $C^*(\mathfrak{L}_2 \cap \mathfrak{L}_\infty)$, then it would extend to a linear isometry $f \to \tilde{\phi}_f$ of $\mathfrak{L}_1$ onto $\mathfrak{L}_\infty^*$ given by: $\tilde{\phi}_f(g) = \int_X fg\,dl$, $f \in \mathfrak{L}_1$, $g \in \mathfrak{L}_\infty$. Using the special linear isometry of $\mathfrak{L}_\infty$ onto $\mathfrak{L}_1^*$ (see [3, 19.31 and 20.20]), one shows easily that $\mathfrak{L}_1$ would have to be reflexive. This is known to be false even for $X = [0, 1]$ and l Lebesgue measure.

4. **The trace-class and two norms.** The trace-class of A possesses two norms, $\| \ \|$ and τ, neither of which is in general complete or an algebra norm. Multiplication in $\tau(A)$ is separately continuous with respect to each norm. There are two relationships between $\| \ \|$ and τ: for any x and y in A,

$$\tau(x^*x) = \|x\|^2 \quad \text{and} \quad \tau(xy) \leq \|x\| \|y\|.$$

These two norms are not in general comparable—Theorem 2.5 shows that there is no constant K such that $\|a\| \leq K\tau(a)$ for all a in $\tau(A)$ unless A is an H^*-algebra after trivial renorming, and the following result show that there is not generally any such reverse inequality.

4.1 Theorem. *The following statements are equivalent*:

(1) *There is a constant* $K > 0$ *such that, for every* a *in* $\tau(A)$, $\tau(a) \leq K\|a\|$.

(2) *There is a constant* $K > 0$ *such that, for every* x *in* A, $\|x\| \leq K\|\bar{L}_x\|$.

(3) *There is a constant* $M > 0$ *such that, for every* a *in* $\tau(A)$, $\tau(a) \leq M\tau(\bar{L}_a)$.

(4) A *is projection bounded from above.*

(5) A *has an identity.*

Proof. If (1) is true then, for any x in A,

$$\|x\|^2 = \tau(x^*x) \leq K\|x^*x\| \leq K\|\bar{L}_x\|\,\|x\|$$

so $\|x\| \leq K\|\bar{L}_x\|$. If (2) holds, then for any a in $\tau(A)$,

$$\tau(a) = \|[a]^{1/2}\|^2 \leq K^2\|\bar{L}_{[a]^{1/2}}\|^2 = K^2\|\bar{L}_{[a]}\| = K^2\|\bar{L}_a\|$$

so (3) is true, since $\tau(\bar{L}_a) = \|\bar{L}_a\|$ by (5) of Theorem 2.4. Suppose now (3) is true; if e is any projection of A, then e is in $\tau(A)$ and $\tau(e) = \|e\|^2$, $\tau(\bar{L}_e) = \|\bar{L}_e\| = 1$, therefore $\|e\| \leq M^{1/2}$. Thus A is projection bounded from above. Suppose now there is a constant $c > 0$ such that $\|p\| \leq c$ for each projection p of A. Let $\{e_\gamma : \gamma \in \Gamma\}$ be any projection base for A. If F is any finite subset of Γ, then $\Sigma_{\gamma \in F}\|e_\gamma\|^2 = \|\Sigma_{\gamma\in F}\, e_\gamma\|^2 \leq c^2$, so $\Sigma_{\gamma\in\Gamma}\|e_\gamma\|^2 \leq c^2$. Thus Γ must be countable, so let the projection base be denoted $\{e_n\}_{n=1}^{\infty}$. Then $\{\Sigma_{n=1}^{m} e_n\}_{m=1}^{\infty}$ is a Cauchy sequence, so it has a limit $e = \Sigma_{n=1}^{\infty} e_n$ in H. Note that $e^* = e$. Moreover e is a bounded element of H: for any y in A,

$$L_e(y) = \bar{R}_y(e) = \lim_{m\to\infty} \bar{R}_y\left(\sum_{n=1}^{m} e_n\right) = \lim_{m\to\infty} \sum_{n=1}^{m} e_n y = y,$$

since $\{e_n\}_{n=1}^{\infty}$ is a projection base for A. Therefore $e \in A$, and clearly $ey = y = ye$ for all y in A. Finally, if A has an identity 1, then for any a in $\tau(A) = A$,

$$\mathrm{tr}\,(a) = \mathrm{tr}\,(a1) = (a,\ 1)$$

(note $1^* = 1$), so $\tau(a) = ([a], 1) \leq \|[a]\|\,\|1\| = \|1\|\,\|a\|$. Therefore (1) is true. This completes the proof of the theorem.

Yood [14, Theorem 4.2] gives other conditions on A equivalent to those in the above theorem. For example, $A = C^*(A)$—that is, $\{\bar{L}_x : x \in A\}$ is closed in the operator norm topology on $\mathcal{B}(H)$.

Having seen that these two norms on the trace-class need not be comparable, we attempt to discover what properties they have in common. To begin, we introduce some orthogonal complementation notation. If $S \subset \tau(A)$, put

$S^+ = \{\xi \in H : (\xi, S) = \{0\}\}$, the orthogonal complement of S in H,

$S^\perp = \{x$ in $A : (x, S) = \{0\}\} = S^+ \cap A$, the orthogonal complement in A,

$S^P = \{a$ in $\tau(A) : (a, S) = \{0\}\} = S^\perp \cap \tau(A)$, the orthogonal complement in $\tau(A)$.

It is a consequence of Proposition 2.7 of [7] that $I^{\perp}$ is dense in I^{+} if I is any one-sided ideal of A.

Much of what now follows was inspired by similar results concerning H^*-algebras in [11]. Many of our statements are formulated for left ideals only, it being understood that the corresponding results for right ideals obtain. Regarding closure terminology, a subset of $\tau(A)$ which is closed in the relative $\| \, \|$ (resp. τ) topology on $\tau(A)$ shall be referred to as "$\| \, \|$-closed" (resp. "τ-closed").

4.2 **Lemma.** *If I is a left ideal of $\tau(A)$, then*

(1) $\bar{I}^A$ (the closure in A of I) is a closed left ideal of A;

(2) I^P is a $\| \, \|$-closed left ideal of $\tau(A)$;

(3) if I is $\| \, \|$-closed, then I is a left ideal of A, and I^P is dense in $I^{\perp}$ in the norm $\| \, \|$.

Proof. (1) and (2) are easily shown using separate continuity of multiplication. If I is $\| \, \|$-closed, then $I = \bar{I}^A \cap \tau(A)$ is a left ideal of A. Let $x \in I^{\perp}$, and let $\{e_\gamma : \gamma \in \Gamma\}$ be any projection base for A. Then each $e_\gamma x \in I^{\perp} \cap \tau(A) = I^P$, since $(e_\gamma x, I) = (x, e_\gamma I) = \{0\}$. Therefore $x = \Sigma_{\gamma \in \Gamma} e_\gamma x \in I^{PA}$.

4.3 **Lemma.** *If I is a $\| \, \|$-closed left ideal of $\tau(A)$, then $I = A\,\bar{I}^A = \{xy : x \in A$ and $y \in \bar{I}^A\}$.*

Proof. Put $M = \{xy : x \in A$ and $y \in \bar{I}^A\}$. Then $M \subset A\,\bar{I}^A \subset \bar{I}^A \cap \tau(A) = I$. Now suppose $a = xy \in I$ for some x and y in A. Since A is orthocomplemented [14, Theorem 2.5] we can write $y = y_1 + y_2$ with y_1 in $\bar{I}^A$ and y_2 in $(\bar{I}^A)^{\perp}$. By Lemma 4.2, $(\bar{I}^A)^{\perp} = I^{\perp} = \overline{I^P}^A$; also $\bar{I}^A$ and $(\bar{I}^A)^{\perp}$ are left ideals of A. Therefore $xy_1 \in \bar{I}^A$ and $xy_2 \in (\bar{I}^A)^{\perp}$. Hence $a - xy_1 = xy_2 \in \bar{I}^A \cap (\bar{I}^A)^{\perp} = (0)$, so $a = xy_1 \in M$.

4.4 **Lemma.** *If $\{e_\gamma : \gamma \in \Gamma\}$ is a projection base for A, then for each a in $\tau(A)$, $a = \Sigma_{\gamma \in \Gamma} a e_\gamma = \Sigma_{\gamma \in \Gamma} e_\gamma a$ (convergence in the τ norm).*

Proof. Write $a = xy$ for some x and y in A. If F is any finite subset of Γ, then

$$\tau\left(a - \sum_{\gamma \in F} a e_\gamma\right) = \tau\left(xy - x \sum_{\gamma \in F} y e_\gamma\right) \leq \|x\| \left\| y - \sum_{\gamma \in F} y e_\gamma \right\|.$$

This shows that $\Sigma_{\gamma \in \Gamma} a e_\gamma$ is summable to a in the norm τ. The other equality is similarly shown.

4.5 **Theorem.** *A left ideal I of $\tau(A)$ is τ-closed if and only if it is $\| \, \|$-closed.*

Proof. Suppose I is τ-closed. We need to show that $I = \overline{I}^A \cap \tau(A)$. If $a \in \overline{I}^A \cap \tau(A)$, then for any $\epsilon > 0$ there is a projection p of A such that $\tau(a - pa) < \epsilon/2$, by Lemma 4.4. Also there is a $b \in I$ with $\|p\|\, \|a - b\| < \epsilon/2$. Then $pb \in I$ and

$$\tau(a - pb) \leq \tau(a - pa) + \tau(pa - pb) \leq \epsilon/2 + \|p\|\|a - b\| < \epsilon.$$

Therefore $a \in I$. Hence I is $\|\ \|$-closed.

Conversely, suppose I is $\|\ \|$-closed. $\tau(A)$ is a dual Hilbert algebra by Theorem 2.2 and Corollary 2.2 of [14], so $I = I^{PP}$. But I^{*P} is a $\|\ \|$-closed right ideal of $\tau(A)$ whose left annihilator is $I^{*P*P} = I^{PP} = I$, which is consequently τ-closed, since the separate τ-continuity of multiplication in $\tau(A)$ forces any left or right annihilator in $\tau(A)$ to be τ-closed.

4.6 **Corollary.** *The $\|\ \|$-closure of any left ideal of $\tau(A)$ is equal to its τ-closure.*

For the definition of an orthocomplemented Hilbert algebra, see [14, Definition 2.3]. In the proof of Theorem 4.5 we used the fact that $\tau(A)$ is a dual Hilbert algebra; it is also orthocomplemented.

4.7 **Theorem.** *If I is any closed left ideal of $\tau(A)$, then $\tau(A) = I \oplus I^P$.*

Proof. Let $J = \overline{I}^A$. Then $A = J \oplus J^\perp$ since A is orthocomplemented [14, Theorem 2.5]. For any $a \in \tau(A)$, write $a = xy$ for some x and y in A. We can write $y = y_1 + y_2$ with $y_1 \in J$, $y_2 \in J^\perp$. Since $J^\perp = I^\perp = \overline{I^P}^A$ by Lemma 4.2, it follows from Lemma 4.3 that $xy_1 \in I$ and $xy_2 \in I^P$. Thus $a = xy_1 + xy_2 \in I \oplus I^P$.

Thus the trace-class of a full Hilbert algebra provides another example of an orthocomplemented Hilbert algebra which is not full (see [14, Example 2.6]).

A simple argument based on the orthocomplementation property in A shows that the trace-class of any closed ideal J of A (which is also a full Hilbert algebra by Theorem 2.7 of [14]) is given by $\tau(J) = J \cap \tau(A)$. Using this we can obtain a characterization of the closed ideals of $\tau(A)$.

4.8 **Theorem.** *If J is a closed ideal of A, then $\tau(J)$ is a closed ideal of $\tau(A)$. Conversely, any closed ideal I of $\tau(A)$ has the form $\tau(J)$ where $J = \overline{I}^A$.*

Proof. If J is a closed ideal of A, then $\tau(J) = J \cap \tau(A)$ is clearly an ideal of $\tau(A)$, and is closed in $\tau(A)$: the closure of $\tau(J)$ means the $\|\ \|$-closure by Corollary 4.6, so the closure of $\tau(J)$ is contained in $J \cap \tau(A) = \tau(J)$.

Suppose now that I is a closed ideal of $\tau(A)$. Then $\overline{I}^A$ is a closed ideal of A, so $\tau(\overline{I}^A) = \overline{I}^A \cap \tau(A) = I$.

BIBLIOGRAPHY

1. J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien* (*Algèbres de von Neumann*), Gauthier-Villars, Paris, 1969.

2. R. Godement, *Théorie des caractères. I. Algèbres unitaires*, Ann. of Math. (2) 59 (1954), 47–62. MR 15, 441.

3. E. Hewitt and K. Stromberg, *Real and abstract analysis. A modern treatment of the theory of functions of a real variable*, Springer-Verlag, New York, 1965. MR 32 #5826.

4. B. E. Johnson, *An introduction to the theory of centralizers*, Proc. London Math. Soc. (3) 14 (1964), 299–320. MR 28 #2450.

5. C. N. Kellogg, *Centralizers and H^*-algebras*, Pacific J. Math. 17 (1966), 121–129. MR 33 #1749.

6. H. Nakano, *Hilbert algebras*, Tôhoku Math. J. (2) 2 (1950), 4–23. MR 12, 837.

7. M. A. Rieffel, *Square-integrable representations of Hilbert algebras*, J. Functional Analysis 3 (1969), 265–300. MR 39 #6094.

8. P. P. Saworotnow and J. C. Friedell, *Trace-class for an arbitrary H^*-algebra*, Proc. Amer. Math. Soc. 26 (1970), 95–100. MR 42 #2304.

9. P. P. Saworotnow, *Trace-class and centralizers of an H^*-algebra*, Proc. Amer. Math. Soc. 26 (1970), 101–104. MR 42 #2305.

10. R. Schatten, *Norm ideals of completely continuous operators*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Heft 27, Springer-Verlag, Berlin, 1960. MR 22 #9878.

11. J. F. Smith, *The p-classes of an H^*-algebra*, Pacific J. Math. (to appear).

12. O. Takenouchi, *On the maximal Hilbert algebra*, Tôhoku Math. J. (2) 3 (1951), 123–131. MR 14, 290.

13. J. G. Wendel, *Left centralizers and isomorphisms of group algebras*, Pacific J. Math. 2 (1952), 251–261. MR 14, 246.

14. B. Yood, *Hilbert algebras as topological algebras* (to appear).

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